A COINCIDENCE FORMULA FOR FOLIATED MANIFOLDS

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Abstract. The main result of the present paper is a coincidence formula for foliated manifolds. To prove this we establish Künneth formula, Poincaré duality and intersection product in the context of tangential de Rham cohomology and homology of tangential currents.

We apply the formula to get a dynamical Lefschetz formula for foliated flows.

1. Introduction

1.1. Consider the category of foliated manifolds with foliated maps. A foliated manifold M is a smooth, i.e. C^∞, manifold without boundary together with a decomposition into connected subsets, called leaves, which is locally trivial. Here local triviality means that each point of M has an open neighborhood U which is diffeomorphic to a Euclidean vector space V such that the induced partition on U corresponds to the partition of V into the cosets of a vector subspace W of V. The quotient V/W is called a local transversal manifold. A foliated map is a smooth map such that leaves are mapped to leaves.

Denote by $\mathcal{T}$ the covariant tangent functor. Attach to a foliated manifold M the tangential subbundle $\mathcal{F}M$ of $\mathcal{T}M$ consisting of vectors which are tangent to the leaves. Since the derivative of a foliated map induces a map of tangential subbundles we get a functor $\mathcal{F}$. Passing to quotients brings the functor $\mathcal{Q} = \mathcal{T}/\mathcal{F}$. For a foliated manifold M the vector bundle $\mathcal{Q}M$ is the transversal bundle of M.

Let M be a foliated manifold. A transversal Riemannian metric on M is a Riemannian metric in $\mathcal{Q}M$ which is locally the pull back of a metric on the local transversal manifold. Next identify $\mathcal{T}M = \mathcal{F}M \oplus \mathcal{Q}M$ by a splitting. Then a bundle-like metric on M is a Riemannian metric in $\mathcal{T}M$ which is the direct sum of a Riemannian metric in $\mathcal{F}M$ and a transversal Riemannian metric on M.

Mimic the proceeding in the classical de Rham theory with the whole tangent bundle replaced by the tangential subbundle to attain to the contravariant functors $\Omega^*_{\mathcal{F}}$ and $\Omega^*_{\mathcal{F},c}$, the tangential differential forms (with compact support). These are functors to nuclear graded differential $\mathbb{R}$-vector spaces. The corresponding cohomology functors $H^*_{\mathcal{F}}$ and $H^*_{\mathcal{F},c}$ are the tangential cohomology and the tangential cohomology with compact support.

Equip the topological dual E' of a topological vector space E with the weak topology. Applying the functor continuous linear forms to tangential differential forms define tangential currents with compact support $\Omega^{\mathcal{F}}_{c,*}$. Similarly, there are the tangential currents $\Omega^{\mathcal{F}}_*$ dual to the tangential differential forms with compact support. In this way, find the covariant functors $H^{\mathcal{F}}_{c,*}$ and $H^{\mathcal{F}}_*$, the tangential homology (with compact support).

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The differentials just introduced are not topological homomorphisms in general. Consequently, it is reasonable to consider the maximal Hausdorff quotients of tangential (co)homology (with compact support). This reduction preserves the functorial properties. The resulting functors are denoted by $\bar{H}^*_{\mathcal{F}}$, $\bar{H}^*_{\mathcal{F},c}$, $\bar{H}^{\mathcal{F}}_{c,*}$ and $\bar{H}^{\mathcal{F}}_*$ and are called reduced tangential (co)homology (with compact support). In particular, get a canonical topological isomorphism

$$\bar{H}^{\mathcal{F}}_*(M) \cong \bar{H}^*_{\mathcal{F},c}(M)'$$

for each foliated manifold M.

Let M be a foliated manifold. Then the zeroth reduced tangential cohomology of M (with compact support) consists of those smooth functions (with compact support) which are constant along the leaves. Now let M carry a transversal Riemannian metric and assume that the tangential subbundle of M is oriented. Let the leaves of M have dimension p. Then a non trivial reduced homology class is defined by integration of tangential p-forms against the transversal volume $\mathrm{vol}_{\mathcal{Q}}$ induced by the metric. This is denoted by $\int_M - \ \mathrm{vol}_{\mathcal{Q}}$.

Now we present the main results of this paper in the setting of the motivating application. In this context our coincidence formula is called a dynamical Lefschetz formula.

1.2. Let the real line $\mathbb{R}$ be foliated by points and equipped with the standard metric and orientation. Let M be a compact connected foliated manifold with a bundle-like metric and oriented tangential subbundle. Let p be the dimension of the leaves of M.

Denote by $- \hat{\otimes} -$ the complete tensor product of nuclear vector spaces. Then we have a Künneth theorem, cf. Section 6.

Theorem 1.1. *The canonical maps*

$$C^\infty(\mathbb{R}_{>0}) \hat{\otimes} \bar{H}^*_{\mathcal{F}}(M) \to \bar{H}^*_{\mathcal{F}}(\mathbb{R}_{>0} \times M)$$

and

$$\bigoplus_\kappa C^\infty_c(\mathbb{R}_{>0}) \hat{\otimes} \big(\bar{H}^{p-\kappa}_{\mathcal{F}}(M) \hat{\otimes} \bar{H}^{\kappa}_{\mathcal{F}}(M)\big) \to \bar{H}^p_{\mathcal{F},c}(\mathbb{R}_{>0} \times M \times M)$$

are topological isomorphisms.

The first statement is similar to a classical result concerning smooth functions with values in a nuclear Fréchet space. The proof of the second involves a tangential Hodge decomposition established by Álvarez López and Kordyukov [1, Corollary 1.3].

Special cases of Poincaré duality take the following shape in this context, cf. Theorem 7.5.

Theorem 1.2. *The maps*

$$\bar{H}^{p-*}_{\mathcal{F}}(M) \to \bar{H}^*_{\mathcal{F}}(M)',$$
$$\omega \mapsto \int_M \omega \wedge - \ \mathrm{vol}_{\mathcal{Q}},$$

and the similar map

$$\bar{H}^p_{\mathcal{F}}(\mathbb{R}_{>0} \times M \times M) \to \bar{H}^p_{\mathcal{F},c}(\mathbb{R}_{>0} \times M \times M)'$$

are injective and have dense image.

For finite dimensional $\bar{H}^*_{\mathcal{F}}(M)$ the first map is an isomorphism. Here the approximation is done by generalizing the convolution of a distribution with a regularizing sequence of smooth functions. More precisely, for the second result we define a sequence of maps

$$R_\nu : \bar{H}^p_{\mathcal{F},c}(\mathbb{R}_{>0} \times M \times M)' \to \bar{H}^p_{\mathcal{F}}(\mathbb{R}_{>0} \times M \times M),$$

such that $\int_{\mathbb{R}_{>0}\times M\times M} R_\nu S \wedge -\ \mathrm{vol}_{\mathcal{Q}}$ converges to S for each continuous linear form $S \in \bar{H}^p_{\mathcal{F},c}(\mathbb{R}_{>0} \times M \times M)'$.

1.3. Let

$$f : \mathbb{R}_{>0} \times M \to M$$

be a foliated map. Then the graph of f is a foliated submanifold of $\mathbb{R}_{>0} \times M \times M$ which is isomorphic to $\mathbb{R}_{>0} \times M$. Moreover, the Künneth theorem allows to define a reduced tangential homology class $[f] \in \bar{H}^{\mathcal{F}}_p(\mathbb{R}_{>0} \times M \times M)$ by the requirement that

$$\langle [f], \omega \otimes \tau \rangle = \int_{\mathbb{R}_{>0}\times M} \omega \wedge f^*\tau\ \mathrm{vol}_{\mathcal{Q}}$$

for $\omega \in \bar{H}^{t,p-\kappa}_c(\mathbb{R}_{>0} \times M)$, $\tau \in \bar{H}^{t,\kappa}(M)$ and $0 \le \kappa \le p$.

In particular, let now

$$\begin{aligned} \phi : \mathbb{R} \times M &\to M, \\ (t,a) &\mapsto \phi^t(a) \end{aligned}$$

be a global foliated flow on M with generated vector field X and let

$$\pi : \mathbb{R}_{>0} \times M \to M$$

be the projection.

Suppose that the graphs Γ of ϕ and $\tilde{\Delta}$ of π intersect foliatedly transversally, i.e.

$$\begin{aligned} \mathcal{T}_{(t,a,a)}\Gamma + \mathcal{T}_{(t,a,a)}\tilde{\Delta} &= \mathcal{T}_{(t,a,a)}(\mathbb{R}_{>0} \times M \times M), \\ \mathcal{F}_{(t,a,a)}\Gamma + \mathcal{F}_{(t,a,a)}\tilde{\Delta} &= \mathcal{F}_{(t,a,a)}(\mathbb{R}_{>0} \times M \times M) \end{aligned}$$

for all $(t,a) \in \mathbb{R}_{>0} \times M$ with $\phi^t(a) = a$. Then the intersection $\Gamma \cap \tilde{\Delta}$ is either empty or a one dimensional submanifold of $\mathbb{R}_{>0} \times M \times M$ which is transversal to the leaves. It consists of components

$$\mathbb{R}_{>0} \times \{a\} \times \{a\},$$

where a is a fixed point of ϕ, and

$$\{\nu l(\gamma)\} \times \big\{ \big(\phi^t(b_\gamma), \phi^t(b_\gamma)\big) \mid t \in \mathbb{R} \big\},$$

$\nu = 1, 2 \ldots$, where γ is a periodic orbit with least positive period $l(\gamma)$ and $b_\gamma \in \gamma$ is arbitrary. For each γ and ν the derivative of $\phi^{\nu l(\gamma)}$ induces an endomorphism

$$\overline{\mathcal{Q}}_{b_\gamma}\phi^{\nu l(\gamma)} : \mathcal{Q}_{b_\gamma}M/\mathbb{R}\cdot\overline{X}_{b_\gamma} \to \mathcal{Q}_{b_\gamma}M/\mathbb{R}\cdot\overline{X}_{b_\gamma}.$$

If we let $\mathrm{pr} : \mathbb{R}_{>0} \times M \times M \to \mathbb{R}_{>0}$ be the projection then from Example 9.4 we get

Theorem 1.3. *The intersection product*

$$[\phi] \bullet [\pi] := \lim_{\nu\to\infty} \langle [\phi], R_\nu[\pi] \wedge - \rangle$$

is defined as a continuous linear form on $\bar{H}^0_{\mathcal{F},c}(\mathbb{R}_{>0} \times M \times M)$*. Its push forward* $\mathrm{pr}_*([\phi] \bullet [\pi])$ *is a distribution on* $\mathbb{R}_{>0}$ *which is a sum of local contributions coming from the fixed points and the periodic orbits of* ϕ*. Explicitly,*

$$\text{(1)} \quad \mathrm{pr}_*([\phi] \bullet [\pi]) = \sum_a \operatorname{sgn}\det(\mathrm{id} - \mathcal{F}_a \phi^{t_a}) \int_{\mathbb{R}_{>0}} \frac{1}{\left|\det\bigl(\mathrm{id} - \mathcal{Q}_a \phi^t\bigr)\right|} \cdot - \, dt$$
$$+ \sum_\gamma l(\gamma) \sum_{\nu=1}^{\infty} \operatorname{sgn}\det(\mathrm{id} - \mathcal{F}_{b_\gamma} \phi^{\nu l(\gamma)}) \frac{1}{\left|\det\bigl(\mathrm{id} - \overline{\mathcal{Q}}_{b_\gamma} \phi^{\nu l(\gamma)}\bigr)\right|} \cdot \delta_{\nu l(\gamma)},$$

where $t_a > 0$ *is arbitrary and* δ_{t_0} *is the Dirac distribution in* t_0*.*

The main step in the proof of this theorem is a direct computation of the approximately defined intersection product.

1.4. Again let ϕ be a global foliated flow on M with generated vector field X and let π and pr be the projections as above. Instead of a transversality condition assume now that the reduced tangential cohomology $\bar{H}^*_{\mathcal{F}}(M)$ of M is finite dimensional. Then the Lefschetz number

$$L(\phi^t) := \sum_\kappa (-1)^\kappa \operatorname{Tr}\bigl(\phi^{t*} : \bar{H}^\kappa_{\mathcal{F}}(M) \to \bar{H}^\kappa_{\mathcal{F}}(M)\bigr)$$

is defined for each t. By the Künneth formula $L(\phi^t)$ defines a smooth function $L(\phi)$ on $\mathbb{R}_{>0}$. In this situation, cf. Section 10, one has

Theorem 1.4. *The intersection product* $[\phi] \bullet [\pi]$ *is defined. Its push forward* $\mathrm{pr}_*([\phi] \bullet [\pi])$ *is the distribution on* $\mathbb{R}_{>0}$ *which is associated with the function* $L(\phi)$*.*

To prove this result, besides the Künneth theorem, the Poincaré duality isomorphism for M is the essential tool.

Now suppose in addition that the graphs of ϕ and π intersect foliatedly transversally. Then we can combine the Theorems 1.3 and 1.4 to get our version of the dynamical Lefschetz formula.

Corollary 1.5. *For each* $t > 0$ *we have*

$$L(\phi^t) = \sum_a \operatorname{sgn}\det(\mathrm{id} - \mathcal{F}_a \phi^{t_a}) \frac{1}{\left|\det\bigl(\mathrm{id} - \mathcal{Q}_a \phi^t\bigr)\right|}$$

and

$$\sum_{t/l(\gamma) \in \mathbb{Z}} l(\gamma) \operatorname{sgn}\det(\mathrm{id} - \mathcal{F}_{b_\gamma} \phi^t) \frac{1}{\left|\det\bigl(\mathrm{id} - \overline{\mathcal{Q}}_{b_\gamma} \phi^t\bigr)\right|} = 0.$$

1.5. If M is foliated by points a formula of Guillemin and Sternberg [12, Chapter VI, p. 311] may be read as a dynamical Lefschetz formula. Here the trace of ϕ^* is defined by a certain pull back process for distributions.

If M is connected, oriented and foliated as one leaf then $\bar{H}^*_{\mathcal{F}}(M) = H^*(M)$ is finite dimensional. By the homotopy property of de Rham cohomology $L(\phi)$ is constant and equal to the Euler characteristic of M. Assume transversal intersection of the graphs in the classical sense. Then the zeroes of X are non degenerate and the Hopf index formula can be interpreted as a dynamical Lefschetz formula.

Apart from these all results on dynamical Lefschetz formulas are concerned with fixed point free flows.

For arbitrary leaf dimension the right hand side of Equation (1) appears if the formula of Guillemin and Sternberg is applied to the exterior bundle of the dual of $\mathcal{F}M$. With varying generality, this is written up by Guillemin [11], Deninger [7] and Deninger, Singhof [8]. The crucial point now is to find conditions which justify to pass from the alternating sum of traces on tangential forms to the alternating sum of traces on tangential cohomology.

Several examples where this passage is possible for foliated manifolds with leaves of codimension one are given in the literature: Guillemin [11] considers the Selberg trace formula. Álvarez López, Kordyukov [2], Deninger, Singhof [8] and Lazarov [14] prove dynamical Lefschetz formulas for foliated manifolds with a bundle-like metric.

A dynamical Lefschetz formula does not necessarily hold for foliated manifolds which do not admit a transversal Riemannian metric. A counterexample is given by Deninger and Singhof in [4].

The significance of dynamical Lefschetz formulas in a wider context is discussed by Deninger [5, 6] and Juhl [13].

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2. Foliated Submanifolds and Transversality

Let M be a foliated manifold. For convenience we write $M^{p,q}$ to indicate that the leaves of M have dimension p and the local transversal manifolds have dimension q. The pair (p,q) is called the foliated dimension of M.

Definition 2.1. A subset $S \subseteq M^{p,q}$ is a *foliated submanifold* of M if for every $a \in S$ there is a foliated chart

$$\phi : U \to U' \times V'$$

around a with $U' \subseteq \mathbb{R}^r \times \mathbb{R}^{p-r}$ and $V' \subseteq \mathbb{R}^s \times \mathbb{R}^{q-s}$, such that

$$\phi(S \cap U) = \big(U' \cap (\mathbb{R}^r \times \{0\})\big) \times \big(V' \cap (\mathbb{R}^s \times \{0\})\big).$$

The notion foliated codimension will be used with the obvious meaning.

Example 2.2. Let $f : M \to N$ be a foliated map. Then the graph Γ_f of f is a closed foliated submanifold of $M \times N$, which is isomorphic to M.

Definition 2.3. Let $f : M \to N$ be a foliated map and let $S \subseteq N$ be a foliated submanifold. The map f is *foliatedly transversal over S* if

$$\mathcal{T}_a f(\mathcal{T}_a M) + \mathcal{T}_{f(a)} S = \mathcal{T}_{f(a)} N, \qquad \mathcal{F}_a f(\mathcal{F}_a M) + \mathcal{F}_{f(a)} S = \mathcal{F}_{f(a)} N$$

for all $a \in f^{-1}(S)$.

Carrying out the obvious modifications in the proof for the smooth case we have the following result concerning transversality.

Proposition 2.4. *Let the foliated map $f : M \to N$ be foliatedly transversal over the foliated submanifold $S \subseteq N$ of foliated codimension (t,u). If $f^{-1}(S) \neq \emptyset$, then*

$$f^{-1}(S) \subseteq M$$

is a foliated submanifold of foliated codimension (t,u). □

3. Foliated Riemannian Connections

A foliated fiber bundle is a fiber bundle in the category of foliated manifolds. We call a foliated fiber bundle transversal if the typical fiber is foliated by points.

Let M be a foliated manifold with foliated tangent bundle

$$\pi : \mathcal{T}M \to M.$$

Suppose that M is equipped with a bundle-like metric denoted by $\langle -,- \rangle$ and identify $\mathcal{T}M$ with the Whitney sum $\mathcal{F}M \perp \mathcal{Q}M$. Then a smooth vector field X decomposes uniquely as

$$X = X_{\mathcal{F}} + X_{\mathcal{Q}},$$

where $X_{\mathcal{F}}$, $X_{\mathcal{Q}}$ resp., is a smooth section of $\mathcal{F}M$, $\mathcal{Q}M$ resp. Moreover, if X is foliated the same is true for $X_{\mathcal{Q}}$.

Let $a \in M$ be a point and let Y be a foliated vector field on a neighborhood U of a. For smooth vector fields X and Z on U write

$$\begin{aligned}\langle \nabla_X Y, Z\rangle :=& \frac{1}{2}\big(X\langle Y_{\mathcal{F}}, Z_{\mathcal{F}}\rangle + Y_{\mathcal{F}}\langle Z_{\mathcal{F}}, X\rangle - Z_{\mathcal{F}}\langle Y_{\mathcal{F}}, X\rangle \\ &+ \langle X, [Z_{\mathcal{F}}, Y_{\mathcal{F}}]\rangle + \langle Y_{\mathcal{F}}, [Z_{\mathcal{F}}, X]\rangle - \langle Z_{\mathcal{F}}, [Y_{\mathcal{F}}, X]\rangle \\ &+ X_{\mathcal{Q}}\langle Y_{\mathcal{Q}}, Z_{\mathcal{Q}}\rangle + Y_{\mathcal{Q}}\langle Z_{\mathcal{Q}}, X_{\mathcal{Q}}\rangle - Z_{\mathcal{Q}}\langle Y_{\mathcal{Q}}, X_{\mathcal{Q}}\rangle \\ &+ \langle X_{\mathcal{Q}}, [Z_{\mathcal{Q}}, Y_{\mathcal{Q}}]\rangle + \langle Y_{\mathcal{Q}}, [Z_{\mathcal{Q}}, X_{\mathcal{Q}}]\rangle - \langle Z_{\mathcal{Q}}, [Y_{\mathcal{Q}}, X_{\mathcal{Q}}]\rangle\big).\end{aligned} \tag{2}$$

Then the assignment $(X, Z) \mapsto \langle \nabla_X Y, Z\rangle$ defines a tensor field of type $(0, 2)$ on U. This fact allows the following

Definition 3.1. Let X be a smooth vector field on U. Then the *foliated covariant derivative of Y in the direction of X* is the unique vector field $\nabla_X Y$ satisfying equation (2) for all vector fields Z on U. If ξ is a tangent vector at a then the *foliated covariant derivative $\nabla_\xi Y$ of Y at a in the direction of ξ* is the value at a of $\nabla_X Y$, where X is any vector field with $X_a = \xi$.

For a vector space V and $v \in V$ let

$$\tau_v : \mathcal{T}_v V \to V$$

be the canonical identification. If now ξ is a tangent vector at a then the vector

$$C(\xi, Y) := \mathcal{T}_a Y(\xi) - \tau_{Y_a}^{-1}(\nabla_\xi Y)$$

in $\mathcal{T}_{Y_a}\mathcal{T}M$ depends only on the value of Y at a. More precisely we get

Proposition 3.2. *The map*

$$\begin{aligned} C : \mathcal{T}M \oplus \mathcal{T}M &\to \mathcal{T}\mathcal{T}M \\ (\xi, z) &\mapsto C(\xi, Y)\end{aligned}$$

where Y is any local foliated vector field taking the value z, is a foliated linear connection on M. □

Definition 3.3. The foliated linear connection of Proposition 3.2 is the *foliated Riemannian connection* associated to the bundle-like metric on M.

The main property of the foliated Riemannian connection is that its parallel transport preserves the metric.

Proposition 3.4. *Let $\gamma : I \to M$ be a path. If β_1 and β_2 are parallel transports along γ relative to the foliated Riemannian connection then $\frac{d}{dt}\langle \beta_1(t), \beta_2(t)\rangle = 0$.* □

Corresponding to the foliated Riemannian connection on M there is the exponential map of its geodesic spray. This is a foliated map

$$\exp : \mathscr{O} \to M,$$

which is defined on an open neighborhood $\mathscr{O}$ of the zero section in $\mathcal{T}M$.

4. Foliated Homomorphisms

Let $E \to M$ and $F \to N$ be transversal vector bundles and denote by F^* the vector bundle dual to F. A foliated homomorphism $(f, \rho) : E \to F$ consists of a foliated map $f : M \to N$ and a global foliated section ρ of $f^*F^* \otimes E$. A foliated homomorphism (f, ρ) is called proper if f is proper.

Example 4.1. Let M be a foliated manifold with a bundle-like metric and let $E \to M$ be a transversal vector bundle with a foliated linear connection

$$C : \mathcal{T}M \oplus E \to \mathcal{T}E.$$

Let $\xi \in \mathscr{O}$ be a point in the domain of the exponential map. Then for each vector $z \in E_{\exp(\xi)}$ there exists a unique parallel transport $\beta_{\xi,z}$ along $\exp(t \cdot \xi)$ with respect to C satisfying $\beta_{\xi,z}(1) = z$. Examining the homogeneous linear differential equation for the parallel transport in local coordinates we reveal that the maps

$$\begin{aligned} P_\xi : E_{\exp(\xi)} &\to E_{\pi(\xi)} \\ z &\mapsto \beta_{\xi,z}(0) \end{aligned}$$

fit together to define a foliated section of $\exp^* E^* \otimes \pi^* E|_{\mathscr{O}}$. Hence we get a foliated homomorphism

$$(\exp, P) : \pi^* E|_{\mathscr{O}} \to E.$$

Example 4.2. Let $\mathrm{pr} : \mathbb{R} \times M \to M$ be the projection, where the real line is foliated by points. A global foliated flow on E is a foliated homomorphism

$$(\phi, \rho) : \mathrm{pr}^* E \to E,$$

where ϕ is a foliated flow on M and, with the obvious notation, ρ satisfies

$$\rho^0_a = \mathrm{id}, \qquad \rho^{s+t}_a = \rho^s_{\phi^t(a)} \circ \rho^t_a$$

for all $s, t \in \mathbb{R}$ and $a \in M$.

5. Tangential (Co)homology

Let $M^{p,q}$ be a foliated manifold and let $E \to M$ be a transversal vector bundle of rank s. The tangential (differential) k-forms $\Omega^k_{\mathcal{F}}(M, E)$ on M with values in E are the smooth sections of $\bigwedge^k \mathcal{F}M^* \otimes E$ equipped with the C^∞-topology. The tangential de Rham complex of M with values in E consists of the direct sum

$$\Omega^*_{\mathcal{F}}(M, E) := \bigoplus_{k \geq 0} \Omega^k_{\mathcal{F}}(M, E)$$

together with the tangential exterior differentiation $d_{\mathcal{F}}$, which is defined locally as follows. Let $x^1, \ldots, y^q$ be foliated coordinates, let $d_{\mathcal{F}} x^i$ be the tangential 1-form which is dual to $\frac{\partial}{\partial x^i}$ and let $\sigma_1, \ldots, \sigma_s$ be a foliated frame for E. Then

$$d_{\mathcal{F}} f := \sum_{i=1}^{p} \frac{\partial f}{\partial x^i} \, d_{\mathcal{F}} x^i$$

for a smooth function f and

$$d_{\mathcal{F}}\omega := \sum_{I,j} d_{\mathcal{F}} a_I^j \wedge d_{\mathcal{F}} x^I \otimes \sigma_j$$

for a tangential form $\omega = \sum_{I,j} a_I^j \, d_{\mathcal{F}} x^I \otimes \sigma_j$.

Let $F \to M$ be a second transversal vector bundle. If $\omega \in \Omega^k_{\mathcal{F}}(M, E)$ and $\tau \in \Omega^l_{\mathcal{F}}(M, F)$ are given then there is the equality

$$d_{\mathcal{F}}(\omega \wedge \tau) = (d_{\mathcal{F}}\omega) \wedge \tau + (-1)^k \omega \wedge (d_{\mathcal{F}}\tau)$$

of tangential forms with values in $E \otimes F$.

For each compact subset $K \subseteq M$ we denote by $\Omega^k_{\mathcal{F}}(M, E; K)$ the closed vector subspace consisting of tangential k-forms with support contained in K. The space $\Omega^k_{\mathcal{F},c}(M, E)$ of all tangential k-forms with compact support carries its LF-topology. Define the pull back of a tangential form in the obvious way to get contravariant functors $\Omega^*_{\mathcal{F}}$ and $\Omega^*_{\mathcal{F},c}$ from transversal vector bundles with (proper) foliated homomorphisms to nuclear graded differential $\mathbb{R}$-vector spaces. The cohomology functors $H^*_{\mathcal{F}}$ and $H^*_{\mathcal{F},c}$ from transversal vector bundles to topological graded $\mathbb{R}$-vector spaces corresponding to tangential differential forms are the tangential cohomology and the tangential cohomology with compact support.

Let the real line $\mathbb{R}$ be foliated as one leaf and let $\mathrm{pr}_M : M \times \mathbb{R} \to M$ be the projection. Then we have the Poincaré lemma for tangential cohomology, cf. [10], i.e. pr_M^* defines an isomorphism

$$H^*_{\mathcal{F}}(M, E) \cong H^*_{\mathcal{F}}(M \times \mathbb{R}, \mathrm{pr}_M^* E).$$

This implies that tangential cohomology is invariant under certain homotopies which we call tangential.

Let $\pi : V \to M$ be a foliated vector bundle of rank r the typical fiber of which is foliated as one leaf. A tangential form $\omega \in \Omega^*_{\mathcal{F}}(V)$ has compact support in the vertical direction if for each compact $K \subseteq M$ the set $\operatorname{Supp}\omega \cap \pi^{-1}(K)$ is compact. The resulting subcomplex is denoted by $\Omega^*_{\mathcal{F},cv}(V)$, the cohomology $H^*_{\mathcal{F},cv}(V)$ of this complex is the tangential cohomology of V with compact support in the vertical direction. Assume that V is oriented. Then integration along the fiber π_* induces the Thom isomorphism, cf. [15],

$$H^*_{\mathcal{F},cv}(V) \cong H^{*-r}_{\mathcal{F}}(M).$$

The cohomology class $\Phi_{\mathcal{F}}(V) := \pi_*^{-1}(1)$ in $H^r_{\mathcal{F},cv}(V)$ is the tangential Thom class of V. Now fix a Riemannian metric in V. Then for $\epsilon > 0$ there exists a representative of the tangential Thom class with support contained in

$$V(\epsilon) := \{v \in V \mid \|v\| < \epsilon\}.$$

A tangential E-current of dimension k on M is a continuous linear form on $\Omega^k_{\mathcal{F},c}(M, E)$. The vector space of all tangential E-currents of dimension k on M will be denoted by $\Omega_k^{\mathcal{F}}(M, E)$ and equipped with the weak topology. Transposition allows to define the tangential boundary map $b_{\mathcal{F}}$ and the push forward by a proper foliated homomorphism. Hence, the tangential currents become a covariant functor from transversal vector bundles with proper foliated homomorphisms to topological graded codifferential $\mathbb{R}$-vector spaces. The homology functor $H_*^{\mathcal{F}}$ associated with the tangential currents is the tangential homology. Similarly, the tangential currents with compact support $\Omega^{\mathcal{F}}_{c,*}$ are dual to $\Omega^*_{\mathcal{F}}$ and define tangential homology with

compact support $H^{\mathcal{F}}_{c,*}$. We always think of the trivial bundle $E=\mathbb{R}$ if the E is omitted in the notation.

Example 5.1. Let $M^{p,q}$ be a foliated manifold with a transversal Riemannian metric and corresponding transversal volume $\mathrm{vol}_{\mathcal{Q}}$. Let $\mathcal{F}M$ be oriented. Then real valued tangential p-forms are identified with densities and integration defines the closed tangential current $\int_M - \mathrm{vol}_{\mathcal{Q}}$ of dimension p on M. Moreover, the assignment $\omega \mapsto \int_M \omega \wedge - \mathrm{vol}_{\mathcal{Q}}$ induces continuous maps

$$H^{p-k}_{\mathcal{F},c}(M,E^*) \to H^{\mathcal{F}}_{c,k}(M,E), \qquad H^{p-k}_{\mathcal{F}}(M,E^*) \to H^{\mathcal{F}}_{k}(M,E).$$

Since tangential exterior differentiation is not a topological homomorphism in general, we consider the reduced tangential (co-)homology (with compact support) which is the maximal Hausdorff quotient of the corresponding (co-)homology space. By continuity of pull back we get new functors, denoted by $\bar{H}^*_{\mathcal{F}}$, $\bar{H}^*_{\mathcal{F},c}$, $\bar{H}^{\mathcal{F}}_{c,*}$ and $\bar{H}^{\mathcal{F}}_{*}$. Use the Hahn-Banach theorem and the theorem of bipolars to prove

Proposition 5.2. *The canonical maps*

$$\bar{H}^{\mathcal{F}}_{k}(M,E) \to \bar{H}^{k}_{\mathcal{F},c}(M,E)', \qquad \bar{H}^{\mathcal{F}}_{c,k}(M,E) \to \bar{H}^{k}_{\mathcal{F}}(M,E)'$$

are topological isomorphisms. □

6. Künneth Maps

Let $E \to M^{p,q}$ and $F \to N^{r,s}$ be transversal vector bundles and let $E \boxtimes F$ be the external tensor product. Consider the bilinear map

$$- \otimes - : \Omega^*_{\mathcal{F}}(M,E) \times \Omega^*_{\mathcal{F}}(N,F) \to \Omega^*_{\mathcal{F}}(M\times N, E\boxtimes F),$$
$$\omega \otimes \tau := \mathrm{pr}_M^* \omega \wedge \mathrm{pr}_N^* \tau,$$

where pr_M and pr_N are the projections. A bilinear map

$$- \otimes - : \Omega^{\mathcal{F}}_{k}(M,E) \times \Omega^{\mathcal{F}}_{l}(N,F) \to \Omega^{\mathcal{F}}_{k+l}(M\times N, E\boxtimes F)$$

is defined by the requirement that

$$\langle S\otimes T, \omega\otimes\tau\rangle = (-1)^{k(r-l)}\langle S,\omega\rangle\langle T,\tau\rangle$$

for all $\omega \in \Omega^k_{\mathcal{F},c}(M,E)$ and $\tau \in \Omega^l_{\mathcal{F},c}(N,F)$. Both maps restrict to compact supports and descend to reduced (co)homology. Moreover, by linear disjointness with respect to $-\otimes-$ the induced maps

$$\bar{H}^*_{\mathcal{F}(,c)}(M,E) \otimes \bar{H}^*_{\mathcal{F}(,c)}(N,F) \to \bar{H}^*_{\mathcal{F}(,c)}(M\times N, E\boxtimes F),$$
$$\bar{H}^{\mathcal{F}}_{(c,)*}(M,E) \otimes \bar{H}^{\mathcal{F}}_{(c,)*}(N,F) \to \bar{H}^{\mathcal{F}}_{(c,)*}(M\times N, E\boxtimes F)$$

are one-to-one. In special cases we get more precise results for the nuclear cohomology spaces.

Example 6.1. Let M be foliated by points. Similar to Theorems 40.1 and 44.1 of [16] one proves the topological isomorphisms

$$\Gamma_{(c)}(M,E) \,\hat{\otimes}\, \bar{H}^*_{\mathcal{F}(,c)}(N,F) \cong \bar{H}^*_{\mathcal{F}(,c)}(M\times N, E\boxtimes F).$$

Example 6.2. Let M be a compact foliated manifold with bundle-like metric and oriented tangential subbundle. Suppose we are given a transversal Riemannian metric in E, i.e. the corresponding section of $E^* \otimes E^*$ is foliated. Define the tangential star operator

$$\star_{\mathcal{F}} : \Omega^k_{\mathcal{F}}(M, E) \to \Omega^{p-k}_{\mathcal{F}}(M, E)$$

by the requirement that $\omega \wedge \star_{\mathcal{F}} \tau = \langle \omega, \tau \rangle \ \mathrm{vol}_{\mathcal{F}}$ for all $\omega \in \Omega^k_{\mathcal{F}}(M, E)$ and let

$$\delta_{\mathcal{F}} := (-1)^{p \cdot * + 1} \star_{\mathcal{F}} d_{\mathcal{F}} \star_{\mathcal{F}} : \Omega^{*+1}_{\mathcal{F}}(M, E) \to \Omega^*_{\mathcal{F}}(M, E).$$

The tangential Laplacian of E is the operator $\Delta_{\mathcal{F}} := d_{\mathcal{F}} \circ \delta_{\mathcal{F}} + \delta_{\mathcal{F}} \circ d_{\mathcal{F}}$ and we write

$$\mathscr{H}^k_{\mathcal{F}}(M, E) := \ker \Delta_{\mathcal{F}} \cap \Omega^k_{\mathcal{F}}(M, E)$$

for the tangentially harmonic k-forms. By adjointness with respect to the L^2-inner product $\int_M - \wedge \star_{\mathcal{F}} - \ \mathrm{vol}_{\mathcal{Q}}$ we find $\mathscr{H}_{\mathcal{F}}(M, E) = \ker d_{\mathcal{F}} \cap \ker \delta_{\mathcal{F}}$ hence a map

$$\mathscr{H}^*_{\mathcal{F}}(M, E) \to \bar{H}^*_{\mathcal{F}}(M, E),$$

which is an inclusion since $\star_{\mathcal{F}} \circ \Delta_{\mathcal{F}} = \Delta_{\mathcal{F}} \circ \star_{\mathcal{F}}$. It is a consequence of the tangential Hodge decomposition

$$\Omega^*_{\mathcal{F}}(M, E) = \mathscr{H}^*_{\mathcal{F}}(M, E) \oplus \overline{\operatorname{im} d_{\mathcal{F}}} \oplus \overline{\operatorname{im} \delta_{\mathcal{F}}},$$

cf. [1, Corollary 1.3], that this is a topological isomorphism. Assume that N is a compact foliated manifold with bundle-like metric and oriented tangential subbundle, too. Let $F \to N$ be equipped with a transversal Riemannian metric. Collecting the signs correctly we find the commutativity of the diagram

$$\begin{array}{ccc} \Omega^*_{\mathcal{F}}(M, E) \otimes \Omega^*_{\mathcal{F}}(N, F) & \xrightarrow{\Delta^M_{\mathcal{F}} \otimes \mathrm{id} + \mathrm{id} \otimes \Delta^N_{\mathcal{F}}} & \Omega^*_{\mathcal{F}}(M, E) \otimes \Omega^*_{\mathcal{F}}(N, F) \\ \downarrow & & \downarrow \\ \Omega^*_{\mathcal{F}}(M \times N, E \boxtimes F) & \xrightarrow{\Delta^{M \times N}_{\mathcal{F}}} & \Omega^*_{\mathcal{F}}(M \times N, E \boxtimes F) \end{array} . \tag{3}$$

Now consider the Hilbert spaces obtained by completion of differential forms with respect to the L^2-product, which are denoted by a prefixed L^2. By [3, Theorem 2.2], the tangential Laplacians define essentially selfadjoint unbounded operators, so let $E^M(t)$, $E^N(t)$ resp., be the spectral families of $\Delta^M_{\mathcal{F}}$, $\Delta^N_{\mathcal{F}}$ resp. The operator $\Delta^M_{\mathcal{F}} \otimes \mathrm{id} + \mathrm{id} \otimes \Delta^N_{\mathcal{F}}$ is essentially selfadjoint in the complete tensor product $L^2\Omega^*_{\mathcal{F}}(M, E) \hat{\otimes} L^2\Omega^*_{\mathcal{F}}(N, F)$ of Hilbert spaces. Its spectral family is given by, cf. [17, proof of Theorem 8.34],

$$E(t) = G(\{z \in \mathbb{C} \mid \Re(z) + \Im(z) \le t\}),$$

where G is the complex spectral family defined by

$$G(t + is) := E^M(t) \hat{\otimes} \mathrm{id} \circ \mathrm{id} \hat{\otimes} E^N(s).$$

The isomorphism

$$L^2\Omega^*_{\mathcal{F}}(M, E) \hat{\otimes} L^2\Omega^*_{\mathcal{F}}(N, F) \cong L^2\Omega^*_{\mathcal{F}}(M \times N, E \boxtimes F)$$

together with the diagram (3) allows to identify E as the spectral family of $\Delta_{\mathcal{F}}^{M\times N}$. Since the tangential Laplacians are nonnegative we get a commutative diagram

$$\begin{array}{ccc} L^2\Omega^*_{\mathcal{F}}(M,E)\,\hat{\otimes}\,L^2\Omega^*_{\mathcal{F}}(N,F) & \xrightarrow{E^M(0)\,\hat{\otimes}\,E^N(0)} & L^2\Omega^*_{\mathcal{F}}(M,E)\,\hat{\otimes}\,L^2\Omega^*_{\mathcal{F}}(N,F) \\ \downarrow & & \downarrow \\ L^2\Omega^*_{\mathcal{F}}(M\times N, E\boxtimes F) & \xrightarrow{E(0)} & L^2\Omega^*_{\mathcal{F}}(M\times N, E\boxtimes F) \end{array}.$$

By the decomposition of tangential forms there is an induced diagram

$$\begin{array}{ccc} \Omega^*_{\mathcal{F}}(M,E)\,\hat{\otimes}\,\Omega^*_{\mathcal{F}}(N,F) & \xrightarrow{\Pi^M\,\hat{\otimes}\,\Pi^N} & \mathscr{H}^*_{\mathcal{F}}(M,E)\,\hat{\otimes}\,\mathscr{H}^*_{\mathcal{F}}(N,F) \\ \downarrow & & \downarrow \\ \Omega^*_{\mathcal{F}}(M\times N, E\boxtimes F) & \xrightarrow{\Pi} & \mathscr{H}^*_{\mathcal{F}}(M\times N, E\boxtimes F) \end{array},$$

and we can replace $\mathscr{H}$ by $\bar{H}$ to find an isomorphism

$$\bar{H}^*_{\mathcal{F}}(M,E)\,\hat{\otimes}\,\bar{H}^*_{\mathcal{F}}(N,F) \cong \bar{H}^*_{\mathcal{F}}(M\times N, E\boxtimes F).$$

7. Regularization

Let $M^{p,q}$ be a foliated manifold with a bundle-like metric in the foliated tangent bundle $\pi : \mathcal{T}M \to M$. Denote by $\mathcal{V}\mathcal{T}M$ the vertical subbundle of $\mathcal{T}\mathcal{T}M$ and let $\upsilon_{\mathcal{Q}}$ be the transversal density in $\mathcal{V}\mathcal{T}M$ induced by the identification $\mathcal{T}_{\pi(\xi)}M \cong \mathcal{V}_\xi\mathcal{T}M$. Let $\mathcal{F}M$ be oriented and let $\phi_{\mathcal{F},\nu}$ be a representative of the tangential Thom class of M with support contained in $\mathcal{F}M(1/\nu)$. Let $\mathrm{pr}_1 : \mathcal{T}M \to \mathcal{F}M$ be the projection and define closed p-forms on $\mathcal{T}M$ by

$$\phi_\nu := \frac{\mathrm{pr}_1^*\,\phi_{\mathcal{F},\nu} \wedge \nu^q \varrho(\nu\cdot\|-\|_{\mathcal{Q}})}{\pi^*\pi_*\big(\mathrm{pr}_1^*\,\phi_{\mathcal{F},\nu} \wedge \varrho(\|-\|_{\mathcal{Q}})\,\upsilon_{\mathcal{Q}}\big)},$$

where $\varrho \in C^\infty(\mathbb{R})$ is any nonnegative function with $\varrho(s) = 1$ for $s < 1/3$, $\varrho(s) = 0$ for $s > 2/3$. Consider $M \times N$, $N := M$, and let pr_M and pr_N be the projections. Finally, let $\mathscr{U}$ be an open neighborhood of the zero section in $\mathcal{T}M$, such that $(\pi,\exp)|_{\mathscr{U}}$ is a foliated embedding.

Suppose now we are given a transversal vector bundle $E \to M$ with a foliated linear connection and recall the foliated homomorphism $(\exp, P)$ constructed in Example 4.1. Let $K \subseteq M$ be compact and let $U \supseteq K$ be a relatively compact open subset of M with closure L. If $\omega \in \Omega^k_{\mathcal{F}}(M,E;K)$ is a tangential form, then Proposition 3.4 implies that for large values of ν the support of $\phi_\nu \wedge (\exp,P)^*\omega$ is contained in a compact subset of $\pi^{-1}(L) \cap \mathscr{U}$. Consider this as a form on $M \times N$ to define

$$R'_\nu\omega := \int_N \phi_\nu \wedge (\exp,P)^*\omega \;\mathrm{vol}_{\mathcal{Q}}\,.$$

By compatibility with the differentials, there are induced linear maps

$$R'_\nu : \bar{H}^k_{\mathcal{F}}(M,E;K) \to \bar{H}^k_{\mathcal{F}}(M,E;L).$$

Lemma 7.1. *For each $\omega \in \bar{H}^k_{\mathcal{F}}(M,E;K)$ the sequence $R'_\nu\omega$ converges to $(-1)^{pk}\omega$ in $\bar{H}^k_{\mathcal{F}}(M,E;L)$.*

Proof. Pull back to the tangent bundle and combine a tangential homotopy argument with a familiar estimate. □

Let S be a tangential E-current of dimension k on M then for large ν Proposition 3.4 allows to define

$$R_\nu S := \langle S_M, \phi_\nu \wedge P \rangle$$

as an element of $\Omega^{p-k}_{\mathcal{F}}(U, E^*)$. Again, we get induced maps

$$R_\nu : \bar{H}^{\mathcal{F}}_k(M, E) \to \bar{H}^{p-k}_{\mathcal{F}}(U, E^*).$$

Lemma 7.2. *For each $S \in \bar{H}^{\mathcal{F}}_k(M, E)$ the sequence $\int_M R_\nu S \wedge - \operatorname{vol}_{\mathcal{Q}}$ converges to $(-1)^{pk} S$ in $\bar{H}^k_{\mathcal{F}}(M, E; K)'$.*

Proof. For $\omega \in \bar{H}^k_{\mathcal{F}}(M, E; K)$ we find

$$\int_M R_\nu S \wedge \omega \operatorname{vol}_{\mathcal{Q}} = \langle S, R'_\nu \omega \rangle$$

and the result follows from Lemma 7.1. □

Conversely, let S be a tangential E-current of dimension k with support contained in K. Then for large ν the form $R_\nu S$ has support in L by Proposition 3.4 and consequently we get maps

$$R_\nu : \bar{H}^{\mathcal{F}}_k(M, E; K) \to \bar{H}^{p-k}_{\mathcal{F}}(M, E^*; L).$$

Similar to the above we have

Lemma 7.3. *For $S \in \bar{H}^{\mathcal{F}}_k(M, E; K)$ the sequence $\int_M R_\nu S \wedge - \operatorname{vol}_{\mathcal{Q}}$ converges to $(-1)^{pk} S$ in $\bar{H}^k_{\mathcal{F}}(M, E)'$.* □

This lemma together with Proposition 5.2 yields

Theorem 7.4. *The map $\bar{H}^{p-k}_{\mathcal{F},c}(M, E^*) \to \bar{H}^k_{\mathcal{F}}(M, E)'$ associated with the integration current has dense image.* □

Assume the existence of $\epsilon > 0$ such that $\mathcal{T} M(\epsilon) \subseteq \mathscr{U}$. Then for large ν the above definition yields maps

$$R_\nu : \bar{H}^{\mathcal{F}}_k(M, E) \to \bar{H}^{p-k}_{\mathcal{F}}(M, E^*)$$

and one obtains Poincaré duality.

Theorem 7.5. *For each k the bilinear map*

$$\int_M - \wedge - \operatorname{vol}_{\mathcal{Q}} : \bar{H}^{p-k}_{\mathcal{F}}(M, E^*) \times \bar{H}^k_{\mathcal{F},c}(M, E) \to \mathbb{R}$$

is a dual pairing. □

Example 7.6. If M is compact an $\epsilon > 0$ as above always exists. In the special case where $\bar{H}^k_{\mathcal{F}}(M, E)$ is finite dimensional we get an isomorphism

$$\bar{H}^{p-k}_{\mathcal{F}}(M, E^*) \cong \bar{H}^k_{\mathcal{F}}(M, E)'.$$

8. Product of Homology Classes

Let $M^{p,q}$ be a foliated manifold with a bundle-like metric, let $\mathcal{F}M$ be oriented and re-use the notations of Section 7. Let E and F be transversal vector bundles over M, where F is equipped with a foliated linear connection. Let $S \in \bar{H}_k^{\mathcal{F}}(M,E)$ and $T \in \bar{H}_l^{\mathcal{F}}(M,F)$, $k+l \geq p$, be given then for $K \subseteq M$ compact

$$\langle S, R_\nu T \wedge - \rangle \in \bar{H}_{\mathcal{F}}^{k+l-p}(M, E \otimes F; K)'$$

is defined for large values of ν. If for each cohomology class $\eta \in \bar{H}_{\mathcal{F},c}^{k+l-p}(M, E \otimes F)$ the sequence $\langle S, R_\nu T \wedge \eta \rangle$ converges we make the following

Definition 8.1. The *intersection product* $S \bullet T$ of S and T is the continuous linear form on $\bar{H}_{\mathcal{F},c}^{k+l-p}(M, E \otimes F)$ given by

$$S \bullet T := \lim_{\nu \to \infty} (-1)^{pl} \langle S, R_\nu T \wedge - \rangle.$$

Example 8.2. Let $\omega \in \bar{H}_{\mathcal{F}}^{p-k}(M, E^*)$ and $\tau \in \bar{H}_{\mathcal{F}}^{p-l}(M, F^*)$ be given, then the intersection product of the corresponding homology classes is defined and

$$\Big(\int_M \omega \wedge - \ \mathrm{vol}_{\mathcal{Q}} \Big) \bullet \Big(\int_M \tau \wedge - \ \mathrm{vol}_{\mathcal{Q}} \Big) = \int_M (\omega \wedge \tau) \wedge - \ \mathrm{vol}_{\mathcal{Q}} .$$

Example 8.3. Let the closed foliated submanifolds S and T of M intersect foliatedly transversally, i.e. the inclusion of S is foliatedly transversal over T, and assume $S \cap T \neq \emptyset$. If the leaves of S have dimension k and those of T have dimension l, then $S \cap T$ is a foliated submanifold with leaves of dimension $k+l-p$, cf. Proposition 2.4. The transversal Riemannian metric on M induces a transversal Riemannian metric on each submanifold and we get transversal Riemannian volumes on S, T and $S \cap T$. Assume $\mathcal{F}S$ and $\mathcal{F}T$ are oriented, then there is an intrinsic proceeding [9, (24.13.14)] to deduce an orientation for $\mathcal{F}(S \cap T)$. Consequently the closed foliated submanifolds define tangential homology classes, again denoted by S, T and $S \cap T$. A tangential homotopy argument followed by some estimation shows that the intersection product $S \bullet T$ is defined and

$$S \bullet T = (-1)^{k(p-k)+l(p-l)} h \cdot S \cap T,$$

where $h \in \bar{H}_{\mathcal{F}}^0(S \cap T)$ is the closed function satisfying

$$h \cdot \mathrm{vol}_{\mathcal{Q}}(S \cap T) \otimes \mathrm{vol}_{\mathcal{Q}}(M) = \mathrm{vol}_{\mathcal{Q}}(S) \otimes \mathrm{vol}_{\mathcal{Q}}(T).$$

By definition of the R_ν we have $S \bullet T = 0$ if S and T have empty intersection.

9. Tangential Coincidence

Let the foliated manifolds $M^{p,q}$ and $N^{r,s}$, $p \geq r$, carry bundle-like metrics and let the tangential subbundles be oriented. Let $E \to M$ and $F \to N$ be transversal vector bundles and let

$$(f, \rho) : E \to F, \qquad\qquad (g, \sigma) : E^* \to F^*$$

be foliated homomorphisms such that the graphs Γ_f and Γ_g intersect foliatedly transversally. Give $\mathcal{F}(M \times N)$ the product orientation and $\mathcal{Q}(M \times N)$ the product metric, orient a graph by demanding the graph map to be orientation preserving and $\Gamma_f \cap \Gamma_g$ as in Section 8. Finally, equip $\mathcal{Q}(\Gamma_f \cap \Gamma_g)$ with the metric induced by that on $M \times N$ and consider ρ, σ resp., as global foliated section of $E \boxtimes F^*|_{\Gamma_f}$,

$E^* \boxtimes F|_{\Gamma_g}$ resp. Let $h_f \in \bar{H}^0_{\mathcal{F}}(\Gamma_f)$ and $h_g \in \bar{H}^0_{\mathcal{F}}(\Gamma_g)$ be the closed functions such that

$$\frac{1}{h_f}\ \mathrm{vol}_{\mathcal{Q}}(\Gamma_f) = \mathrm{vol}_{\mathcal{Q}}(M) = \frac{1}{h_g}\ \mathrm{vol}_{\mathcal{Q}}(\Gamma_g)$$

and let $h \in \bar{H}^0_{\mathcal{F}}(\Gamma_f \cap \Gamma_g)$ be defined by

$$h \cdot \mathrm{vol}_{\mathcal{Q}}(\Gamma_f \cap \Gamma_g) \otimes \mathrm{vol}_{\mathcal{Q}}(M \times N) = \mathrm{vol}_{\mathcal{Q}}(\Gamma_f) \otimes \mathrm{vol}_{\mathcal{Q}}(\Gamma_g).$$

Definition 9.1. The *tangential coincidence* of (f,ρ) and (g,σ) is the reduced homology class

$$\mathrm{Coin}_{\mathcal{F}}\big((f,\rho),(g,\sigma)\big) := \int_{\Gamma_f \cap \Gamma_g} \big(\frac{\rho}{h_f} \wedge \frac{\sigma}{h_g}\big) \wedge - h \cdot \mathrm{vol}_{\mathcal{Q}}$$

in $\bar{H}^{\mathcal{F}}_{p-r}(M \times N)$.

Notice that the definition is arranged, such that by Example 8.3

$$\mathrm{Coin}_{\mathcal{F}}\big((f,\rho),(g,\sigma)\big) = \big(\frac{\rho}{h_f} \wedge \Gamma_f\big) \bullet \big(\frac{\sigma}{h_g} \wedge \Gamma_g\big)$$

if the bundles E and F are equipped with foliated linear connections.

Example 9.2. If $E = F$ and $(g,\sigma) = \mathrm{id}$ then $\Gamma_f \cap \Gamma_g$ consists of the images of the fixed points of f in the diagonal and the tangential coincidence is the distribution

$$\sum_{f(a)=a} \mathrm{sgn}\det(\mathrm{id} - \mathcal{F}_a f) \frac{\mathrm{Tr}(\rho_a)}{|\det(\mathrm{id} - \mathcal{Q}_a f)|} \cdot \delta_{(a,a)},$$

where $\delta_{(a,a)}$ is the Dirac distribution at (a,a).

Example 9.3. Let again $E = F$ and $(g,\sigma) = \mathrm{id}$ and assume the foliated manifold M to be compact and foliated by points. Choose any linear connection in E then

$$\int_M \mathrm{Tr}\Big(\Delta^* R_\nu\big(\frac{\rho}{h_f} \wedge \Gamma_f\big)\Big)\ \mathrm{vol}_{\mathcal{Q}} \xrightarrow{\nu \to \infty} \big\langle \mathrm{Coin}_{\mathcal{F}}\big((f,\rho),\mathrm{id}\big), 1\big\rangle$$

with $\Delta := \Gamma_{\mathrm{id}}$ by Example 8.3. The left hand side is the trace of the smoothing operator defined by $R_\nu(\frac{\rho}{h_f} \wedge \Gamma_f)$ and Lemma 7.3 suggests to look at the right hand side as the trace of the operator defined by $\frac{\rho}{h_f} \wedge \Gamma_f$. Since

$$\big\langle \frac{\rho}{h_f} \wedge \Gamma_f, \omega \otimes \tau \big\rangle = \int_M \omega \wedge (f,\rho)^* \tau\ \mathrm{vol}_{\mathcal{Q}}$$

for $\omega \in \mathit{\Gamma}(M, E^*)$ and $\tau \in \mathit{\Gamma}(M, E)$ this operator is the pull back by (f,ρ).

Example 9.4. Let the positive real numbers $\mathbb{R}_{>0}$ be foliated by points and equipped with the standard metric, let $M = \mathbb{R}_{>0} \times N$, let

$$g = \mathrm{pr}_N : \mathbb{R}_{>0} \times N \to N$$

be the projection, let $E = \mathrm{pr}_N^* F$ and let σ be the identity in each fiber. If

$$(f,\rho) = (\phi,\rho)|_M$$

is the restriction of a global foliated flow on F, then the submanifold $\Gamma_f \cap \Gamma_g$ has components

$$\mathbb{R}_{>0} \times \{a\} \times \{a\},$$

where a is a fixed point of ϕ, and

$$\{\nu l(\gamma)\} \times \{(b,b) \in N \times N \mid b \in \gamma\},$$

$\nu = 1, 2\dots$, where γ is a periodic orbit with least positive period $l(\gamma)$. Let $\mathrm{pr}_{\mathbb{R}_{>0}} : M \times N \to \mathbb{R}_{>0}$ be the projection and assume N to be compact then $\mathrm{pr}_{\mathbb{R}_{>0},*}$ maps the tangential coincidence to the distribution on $\mathbb{R}_{>0}$ given by

$$\sum_a \operatorname{sgn}\det(\mathrm{id} - \mathcal{F}_a\phi^{t_a}) \int_{\mathbb{R}_{>0}} \frac{\mathrm{Tr}(\rho_a^t)}{\left|\det\left(\mathrm{id} - \mathcal{Q}_a\phi^t\right)\right|} \cdot - \, dt$$
$$+ \sum_\gamma l(\gamma) \sum_{\nu=1}^{\infty} \operatorname{sgn}\det(\mathrm{id} - \mathcal{F}_{b_\gamma}\phi^{\nu l(\gamma)}) \frac{\mathrm{Tr}\left(\rho_{b_\gamma}^{\nu l(\gamma)}\right)}{\left|\det\left(\mathrm{id} - \overline{\mathcal{Q}}_{b_\gamma}\phi^{\nu l(\gamma)}\right)\right|} \cdot \delta_{\nu l(\gamma)},$$

where $t_a > 0$ and $b_\gamma \in \gamma$ are arbitrary and

$$\overline{\mathcal{Q}}_{b_\gamma}\phi^{\nu l(\gamma)} : \mathcal{Q}_{b_\gamma}N/\mathbb{R}\cdot\overline{X}_{\phi,b_\gamma} \to \mathcal{Q}_{b_\gamma}N/\mathbb{R}\cdot\overline{X}_{\phi,b_\gamma}$$

is the endomorphism induced by $\mathcal{Q}_{b_\gamma}\phi^{\nu l(\gamma)}$.

10. Coincidence Formula

Let M_1 be a foliated manifold which is foliated by points and let M_2^{p,q_2} and $N^{p,s}$ be compact foliated manifolds with bundle-like metrics and oriented tangential subbundles. Let $E_2 \to M_2$ and $F \to N$ be transversal vector bundles with foliated linear connections and transversal Riemannian metrics, identify the bundles with their duals, write $M := M_1 \times M_2$ and E for the bundle $\mathrm{pr}_{M_2}^* E_2$ over M. Let $(f, \rho) : E \to F$ be a foliated homomorphism. Now assume the reduced cohomology spaces $\bar{H}^*_{\mathcal{F}}(M_2, E_2)$ and $\bar{H}^*_{\mathcal{F}}(N, F)$ to be of finite dimension. Then we can define a foliated homomorphism of trivial transversal vector bundles

$$(f, \rho)^\kappa : M_1 \times \bar{H}^\kappa_{\mathcal{F}}(M_2, E_2) \to \{\odot\} \times \bar{H}^\kappa_{\mathcal{F}}(N, F)$$

by giving the section

$$(f, \rho)^\kappa_a := \left\{ \begin{aligned} \bar{H}^\kappa_{\mathcal{F}}(N, F) &\to \bar{H}^\kappa_{\mathcal{F}}(M_2, E_2) \\ \tau &\mapsto j_a^* \circ (f, \rho)^*\tau \end{aligned} \right\},$$

where $j_a : M_2 \to M$ is the inclusion map opposite a. This section is foliated, i.e. smooth, by the Künneth theorem

$$C^\infty(M_1) \otimes \bar{H}^*_{\mathcal{F}}(M_2, E_2) \cong \bar{H}^*_{\mathcal{F}}(M, E).$$

Now let $(g, \sigma) : E \to F$ be a second foliated homomorphism such that the graphs Γ_f and Γ_g intersect foliatedly transversally. With the duality of Example 7.6 we consider $(g, \sigma)^{p-\kappa}$ as a foliated homomorphism

$$(g, \sigma)^{p-\kappa} : M_1 \times \bar{H}^\kappa_{\mathcal{F}}(M_2, E_2)^* \to \{\odot\} \times \bar{H}^\kappa_{\mathcal{F}}(N, F)^*.$$

Having chosen a Riemannian metric on M_1 the tangential coincidence

$$\mathrm{Coin}_{\mathcal{F}}\left((f, \rho)^\kappa, (g, \sigma)^{p-\kappa}\right) = \int_{M_1} \left((f, \rho)^\kappa \wedge (g, \sigma)^{p-\kappa}\right) \cdot - \, \mathrm{vol}_{\mathcal{Q}} \tag{4}$$

is a distribution on $M_1 \times \{\odot\} \cong M_1$, which is associated with a function.

Theorem 10.1. *Let $\mathrm{pr}_{M_1} : M \times N \to M_1$ be the projection, then*

$$\mathrm{pr}_{M_1,*}\mathrm{Coin}_{\mathcal{F}}\left((f, \rho), (g, \sigma)\right) = \sum_\kappa (-1)^\kappa \,\mathrm{Coin}_{\mathcal{F}}\left((f, \rho)^\kappa, (g, \sigma)^{p-\kappa}\right).$$

In particular, the left hand side is the distribution associated with a function on M_1.

Proof. For explicit calculation let $(\omega_1^\kappa, \dots, \omega_{\alpha_\kappa}^\kappa)$ be a basis of $\bar{H}_{\mathcal{F}}^\kappa(M_2, E_2)$ and let $(\omega^{\kappa,1}, \dots, \omega^{\kappa,\alpha_\kappa})$ be the basis of $\bar{H}_{\mathcal{F}}^{p-\kappa}(M_2, E_2)$ satisfying

$$\int_{M_2} \omega^{\kappa,i_\kappa} \wedge \omega_{j_\kappa}^\kappa \ \mathrm{vol}_{\mathcal{Q}} = \delta_{j_\kappa}^{i_\kappa},$$

cf. Example 7.6. Similarly, let $(\tau_1^\kappa, \dots, \tau_{\beta_\kappa}^\kappa)$ and $(\tau^{\kappa,1}, \dots, \tau^{\kappa,\beta_\kappa})$ be dual bases of $\bar{H}_{\mathcal{F}}^\kappa(N, F)$ and $\bar{H}_{\mathcal{F}}^{p-\kappa}(N, F)$ with

$$\int_N \tau^{\kappa,k_\kappa} \wedge \tau_{l_\kappa}^\kappa \ \mathrm{vol}_{\mathcal{Q}} = \delta_{l_\kappa}^{k_\kappa}.$$

The functions $x_{l_\kappa}^{\kappa,j_\kappa} \in C^\infty(M_1)$ should be the coefficients in the representation

$$(f,\rho)^* \tau_{l_\kappa}^\kappa = \sum_{j_\kappa} x_{l_\kappa}^{\kappa,j_\kappa} \otimes \omega_{j_\kappa}^\kappa$$

and we let $y_{i_\kappa}^{\kappa,k_\kappa} \in C^\infty(M_1)$ be defined by

$$(g,\sigma)^* \tau^{\kappa,k_\kappa} = \sum_{i_\kappa} y_{i_\kappa}^{\kappa,k_\kappa} \otimes \omega^{\kappa,i_\kappa}.$$

Then we get

$$(f,\rho)^\kappa \wedge (g,\sigma)^{p-\kappa} = \sum_{i_\kappa,k_\kappa} x_{k_\kappa}^{\kappa,i_\kappa} y_{i_\kappa}^{\kappa,k_\kappa}. \tag{5}$$

By the Künneth isomorphism

$$\bigoplus_\kappa C_c^\infty(M_1) \otimes \bar{H}_{\mathcal{F}}^{p-\kappa}(M_2, E_2) \otimes \bar{H}_{\mathcal{F}}^\kappa(N, F) \to \bar{H}_{\mathcal{F},c}^p(M \times N, E \boxtimes F)$$

we find the equations

$$\frac{\rho}{h_f} \wedge \Gamma_f = \int_{M\times N} \sum_\kappa \sum_{j_\kappa,k_\kappa} (-1)^{p(p-\kappa)} x_{k_\kappa}^{\kappa,j_\kappa} \otimes \omega_{j_\kappa}^\kappa \otimes \tau^{\kappa,k_\kappa} \wedge - \ \mathrm{vol}_{\mathcal{Q}},$$
$$\frac{\sigma}{h_g} \wedge \Gamma_g = \int_{M\times N} \sum_\kappa \sum_{i_\kappa,l_\kappa} (-1)^\kappa y_{i_\kappa}^{\kappa,l_\kappa} \otimes \omega^{\kappa,i_\kappa} \otimes \tau_{l_\kappa}^\kappa \wedge - \ \mathrm{vol}_{\mathcal{Q}}$$

of reduced tangential homology classes, where the functions h_f and h_g are as in Definition 9.1. Example 8.2 shows that

$$\mathrm{pr}_{M_1,*} \operatorname{Coin}_{\mathcal{F}}\big((f,\rho),(g,\sigma)\big) = \sum_\kappa (-1)^\kappa \int_{M_1} \sum_{i_\kappa,k_\kappa} x_{k_\kappa}^{\kappa,i_\kappa} y_{i_\kappa}^{\kappa,k_\kappa} \cdot - \ \mathrm{vol}_{\mathcal{Q}}\,. \tag{6}$$

Now combine the equations (6), (5) and (4) to obtain the result. □

Remark 10.2. If we drop the assumption, that the graphs intersect foliatedly transversally, the equality

$$\mathrm{pr}_{M_1,*}\Big[\big(\frac{\rho}{h_f} \wedge \Gamma_f\big) \bullet \big(\frac{\sigma}{h_g} \wedge \Gamma_g\big)\Big] = \sum_\kappa (-1)^\kappa \operatorname{Coin}_{\mathcal{F}}\big((f,\rho)^\kappa, (g,\sigma)^{p-\kappa}\big)$$

is still valid.

Example 10.3. If $M_1 = \{\odot\}$, $E_2 = F$, $(f_2, \rho_2) = \mathrm{id}$ and $r = 0$ we recover from Example 9.2 the classical Lefschetz trace formula

$$\sum_{f(a)=a} \operatorname{sgn}\det(\mathrm{id} - \mathcal{T}_a f) \operatorname{Tr}(\rho_a) = \sum_\kappa (-1)^\kappa \operatorname{Tr}\big((f,\rho)^\kappa\big).$$

Example 10.4. Let (f, ρ) be the restriction to $\mathbb{R}_{>0} \times N$ of a global foliated flow and let (g, σ) be the projection as in Example 9.4. Then with the notation introduced above for any $t > 0$ we have

$$\sum_a \operatorname{sgn}\det(\mathrm{id} - \mathcal{F}_a \phi^{t_a}) \frac{\operatorname{Tr}(\rho_a^t)}{\left|\det\left(\mathrm{id} - \mathcal{Q}_a \phi^t\right)\right|} = \sum_\kappa (-1)^\kappa \operatorname{Tr}\left((\phi^t, \rho^t)^* : \bar{H}^\kappa_{\mathcal{F}}(N) \to \bar{H}^\kappa_{\mathcal{F}}(N)\right)$$

and

$$\sum_{t/l(\gamma) \in \mathbb{Z}} l(\gamma) \operatorname{sgn}\det(\mathrm{id} - \mathcal{F}_{b_\gamma} \phi^{\nu l(\gamma)}) \frac{\operatorname{Tr}(\rho^t_{b_\gamma})}{\left|\det\left(\mathrm{id} - \overline{\mathcal{Q}}_{b_\gamma} \phi^t\right)\right|} = 0.$$

References

[1] Jesús A. Álvarez López and Yuri A. Kordyukov. Long time behavior of leafwise heat flow for Riemannian foliations. *Compositio Math.*, 125(2):129–153, 2001.

[2] Jesús A. Álvarez López and Yuri A. Kordyukov. Distributional Betti numbers of transitive foliations of codimension one. In *Foliations: geometry and dynamics (Warsaw, 2000)*, pages 159–183. World Sci. Publishing, River Edge, NJ, 2002.

[3] Paul R. Chernoff. Essential self-adjointness of powers of generators of hyperbolic equations. *J. Functional Analysis*, 12:401–414, 1973.

[4] C. Deninger and W. Singhof. A counterexample to smooth leafwise Hodge decomposition for general foliations and to a type of dynamical trace formulas. *Ann. Inst. Fourier (Grenoble)*, 51(1):209–219, 2001.

[5] Christopher Deninger. Some analogies between number theory and dynamical systems on foliated spaces. In *Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998)*, number Extra Vol. I, pages 163–186 (electronic), 1998.

[6] Christopher Deninger. On dynamical systems and their possible significance for arithmetic geometry. In *Regulators in analysis, geometry and number theory*, pages 29–87. Birkhäuser Boston, Boston, MA, 2000.

[7] Christopher Deninger. Number theory and dynamical systems on foliated spaces. *Jahresber. Deutsch. Math.-Verein.*, 103(3):79–100, 2001.

[8] Christopher Deninger and Wilhelm Singhof. A note on dynamical trace formulas. In *Dynamical, spectral, and arithmetic zeta functions (San Antonio, TX, 1999)*, volume 290 of *Contemp. Math.*, pages 41–55. Amer. Math. Soc., Providence, RI, 2001.

[9] J. Dieudonné. *Éléments d'analyse. Tome IX. Chapitre XXIV.* Gauthier-Villars, Paris, 1982.

[10] Aziz El Kacimi-Alaoui. Sur la cohomologie feuilletée. *Compositio Math.*, 49(2):195–215, 1983.

[11] Victor Guillemin. Lectures on spectral theory of elliptic operators. *Duke Math. J.*, 44(3):485–517, 1977.

[12] Victor Guillemin and Shlomo Sternberg. *Geometric asymptotics.* American Mathematical Society, Providence, R.I., 1977. Mathematical Surveys, No. 14.

[13] Andreas Juhl. *Cohomological theory of dynamical zeta functions*, volume 194 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2001.

[14] C. Lazarov. Transverse index and periodic orbits. *Geom. Funct. Anal.*, 10(1):124–159, 2000.

[15] Calvin C. Moore and Claude Schochet. *Global analysis on foliated spaces.* Springer-Verlag, New York, 1988. With appendices by S. Hurder, Moore, Schochet and Robert J. Zimmer.

[16] François Trèves. *Topological vector spaces, distributions and kernels.* Academic Press, New York, 1967.

[17] Joachim Weidmann. *Linear operators in Hilbert spaces.* Springer-Verlag, New York, 1980. Translated from the German by Joseph Szücs.

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